

ALMOST SIMPLICIAL POLYTOPES I. THE LOWER AND UPPER BOUND THEOREMS

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ABSTRACT. We study n -vertex d -dimensional polytopes with at most one nonsimplex facet with, say, $d + s$ vertices, called *almost simplicial polytopes*. We provide tight lower and upper bound theorems for these polytopes as functions of d, n and s , thus generalizing the classical Lower Bound Theorem by Barnette and Upper Bound Theorem by McMullen, which treat the case $s = 0$. We characterize the minimizers and provide examples of maximizers, for any d .

1. INTRODUCTION

In 1970 McMullen [17] proved the Upper Bound Theorem (UBT) for *simplicial polytopes*, polytopes with each facet being a simplex, while between 1971 and 1973 Barnette [4, 5] proved the Lower Bound Theorem (LBT) for the same polytopes. Both results are major achievements in the combinatorial theory of polytopes; see, e.g., the books [12, 25] for further details and discussion.

These results can be phrased as follows: let $C(d, n)$ (resp. $S(d, n)$) denote a cyclic (resp. stacked) d -polytope on n vertices, and for a polytope P let $f_i(P)$ denote the number of its i -dimensional faces. Then the classical LBT and UBT read as follows.

Theorem 1.1 (Classical LBT and UBT). *For any simplicial d -polytope on n vertices, and any $0 \leq i \leq d - 1$,*

$$f_i(S(d, n)) \leq f_i(P) \leq f_i(C(d, n)).$$

The numbers $f_i(S(d, n))$ and $f_i(C(d, n))$ are explicit known functions of (d, n, i) , to be discussed later.

We generalize the UBT and LBT to the following context: consider a pair (P, F) where P is a polytope, F is a facet of P , and all facets of P different from F are simplices. We call

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such a polytope P an *almost simplicial polytope* (ASP) and a pair (P, F) an ASP-pair. We will be interested only in the combinatorics of P , thus the ASP-pair (P, F) is equivalent to specifying a regular triangulation of F admitting a lifting of its vertices that leaves the vertices of F fixed; we are interested in the simplicial ball $P' := \partial P - \{F\}$.

Let $\mathcal{P}(d, n, s)$ denote the family of d -polytopes P on n -vertices such that (P, F) is an ASP-pair, where F has $d + s$ vertices ($s \geq 0$). Note that $\mathcal{P}(d, n, 0)$ consists of the simplicial d -polytopes on n vertices. In this paper, we define certain polytopes $C(d, n, s), S(d, n, s) \in \mathcal{P}(d, n, s)$, explicitly compute their face numbers, and show the following.

Theorem 1.2 (LBT and UBT for ASP). *For any d, n, s , any polytope $P \in \mathcal{P}(d, n, s)$, and any $0 \leq i \leq d - 1$,*

$$f_i(S(d, n, s)) \leq f_i(P) \leq f_i(C(d, n, s)).$$

Further, the polytopes $P \in \mathcal{P}(d, n, s)$ with $f_i(P) = f_i(S(d, n, s))$ for some $0 \leq i \leq d - 1$ are characterized combinatorially, and satisfy the above equality for all $0 \leq i \leq d - 1$.

The characterization of the equality case above generalizes Kalai's result [13] that equality in the classical LBT holds for some $1 \leq i \leq d - 1$ iff P is stacked. The polytopes $C(d, n, s)$ form an ASP analog of cyclic polytopes and satisfy a combinatorial Gale-evenness type description of their facets.

Billera and Lee [7] considered the notion of polytope pairs. In particular, their results give tight upper and lower bound theorems for the face numbers of simplicial $(d - 1)$ -dimensional balls of the “polytope-antistar” form; that is, balls of the form $\partial Q - v$, where Q is a simplicial d -polytope and v is a vertex of Q that is deleted. These bounds are given as functions of $d, f_0(\partial Q - v), f_0(Q/v)$, where Q/v denotes the vertex figure of v in Q . For an ASP-pair (P, F) , let Q be obtained from P by stacking a pyramid over F with a new vertex v . Then $F \cong Q/v$ and $P' = \partial P - \{F\} = \partial Q - v$. Thus, our balls P' form a subfamily of the balls $\partial Q - v$ considered in [7]. The bounds we obtain in Theorem 1.2 are strictly stronger than those of [7] which apply to all polytope-antistar balls.

Let $f(P) = (1, f_0(P), f_1(P), \dots, f_{d-1}(P))$ denote the f -vector of P , a vector recording the face numbers of P . The following problem naturally arises.

Problem 1.3. *Characterize the pairs of f -vectors $(f(P), f(F))$ for ASP-pairs (P, F) .*

A solution would generalize the well known g -theorem characterizing the face numbers of simplicial polytopes, conjectured by McMullen [18] and proved by Billera-Lee [6] (sufficiency) and Stanley [22] (necessity). We leave this general problem to a future study. However, we give an ASP analogue to the Generalized Lower Bound Theorem [19, 20]; see Theorem 5.1.

The proof of the LBT for ASP and the characterization of the equality cases are based on framework-rigidity arguments (cf. Kalai [13]) and on an adaptation of the well known McMullen-Perles-Walkup reduction (MPW) [13, Sec. 5] to ASP; see Section 3.

The numerical bounds obtained in the UBT for ASP are a special case of a recent result of Adiprasito and Sanyal [2, Thm. 3.10], who proved the bounds for homology balls whose boundary is an induced subcomplex. While their proof relies on machinery from commutative algebra, our proof is elementary and is based on a suitable shelling of P . Further, our construction of maximizers $C(d, n, s)$ is a generalization of cyclic polytopes, based on a suitable variation of the moment curve, and is of independent interest; see Section 4.

2. PRELIMINARIES

For undefined terminology and notation, see [25] for polytopes and complexes, or [13, Sec. 2] for framework rigidity.

2.1. Polytopes and simplicial complexes. The k -dimensional faces of a polyhedral complex Δ are called k -faces, where the empty face has dimension -1 . For a simplicial complex Δ of dimension $d - 1$, the number $f_k(\Delta)$ is then related to the h -numbers $h_k(\Delta) := \sum_{i=0}^k (-1)^{k-i} \binom{d-i}{k-i} f_{i-1}(\Delta)$ by

$$(1) \quad f_{k-1}(\Delta) = \sum_{i=0}^k \binom{d-i}{k-i} h_i(\Delta).$$

The h -vector of Δ , $(\dots, h_k, h_{k+1}, \dots)$, can be considered as an infinite sequence if we let $h_k(\Delta) = 0$ for $k > d$ and $k < 0$. The g -numbers are defined by $g_k(\Delta) = h_k(\Delta) - h_{k-1}(\Delta)$.

For an ASP pair (P, F) , where P is d -dimensional, the following version of the Dehn-Somerville equations applies to the complex $P' = \partial P - \{F\}$.

Proposition 2.1 ([11, Thm. 18.3.6], Dehn-Somerville Equations for P'). *The h -vector of the simplicial $(d - 1)$ -ball P' with boundary ∂F satisfies for $k = 0, \dots, d$*

$$(2) \quad h_k(P') = h_{d-k}(P') + g_k(\partial F).$$

Note that $h_k(P') = 0$ and $h_k(\partial F) = 0$ for $k \geq d$ and $h_{d-1}(\partial F) = 1$.

Let 2^A denote the simplicial complex generated by the set A ; it is a simplex. Sometimes we abbreviate this complex by A , when the context is clear. Say Δ is *pure* if all its maximal faces, called *facets*, have the same dimension, and a pure complex Δ is *shellable* if its facets can be ordered F_1, F_2, \dots such that for each $j > 1$, 2^{F_j} intersects the complex $\cup_{i < j} 2^{F_i}$ in a pure codimension 1 subcomplex of 2^{F_j} . Such an order is called a *shelling order* or *shelling*.

process of Δ . For a shelling order, $2^{F_j} - \cup_{i < j} 2^{F_i}$ has a unique minimal set, called the *restriction face* of F_j , denoted R_j . For *any* shelling of Δ , $h_i(\Delta)$ equals the number of facets in the shelling whose restriction face has size i ; cf. [25, Thm. 8.19]. Note that P' is shellable, by a Bruggesser-Mani line shelling.

The *link* of a face F in Δ is $\text{link}_\Delta(F) := \{T \in \Delta : T \cap F = \emptyset, F \cup T \in \Delta\}$, and its *star*, $\text{star}_\Delta(F)$ is the complex $\cup_{F \subseteq T} 2^T$. Thus, using the *join* operator on simplicial complexes, we obtain $2^F * \text{link}_\Delta(F) = \text{star}_\Delta(F)$. The definition of the star extends to polyhedral complexes. For a vertex v in a polytope Q , its *vertex figure* Q/v is a codimension 1 polytope obtained by intersecting Q with a hyperplane H a bit *below* v , so that v is on one side of H and the other vertices of Q are on the other side. If $\text{star}_Q(v)$ is simplicial then the boundary complex of Q/v coincides with $\text{link}_Q(v)$.

A subcomplex K of Δ is *induced* if it contains all the faces in Δ which only involve vertices in K . Note that, for an ASP-pair (P, F) , ∂F is an induced subcomplex of P' , by convexity.

A polytope is *k-neighborly* if each subset of at most k vertices forms the vertex set of a face. A $\lfloor d/2 \rfloor$ -neighborly d -polytope is simply called *neighborly*. A polytope is *k-simplicial* if each k -face is a simplex.

The underlying set $|\mathcal{C}|$ of a polyhedral complex \mathcal{C} is the point set $\cup_{Q \in \mathcal{C}} Q$ of its geometric realization. A *refinement* (or subdivision) of \mathcal{C} is another polyhedral complex \mathcal{D} such that $|\mathcal{D}| = |\mathcal{C}|$ and for any face $F \in \mathcal{D}$ there exists a face $T \in \mathcal{C}$ such that $|F| \subseteq |T|$.

Let G be a proper face of a polytope Q . A point w is *beyond* G (with respect to Q) if (i) w is not on any hyperplane supporting a facet of Q , (ii) w and the interior of Q lie on different sides of any hyperplane supporting a facet containing G , but (iii) on the same side of every other facet-defining hyperplane which does not contain G . For an ASP-pair (P, F) we will consider the simplicial polytope Q obtained as the convex hull of P and a vertex y beyond F .

A simplicial complex Δ is a *homology sphere* (over a fixed field \mathbf{k}) if for any face $F \in \Delta$, the homology groups $H_i(\text{link}_\Delta(F); \mathbf{k}) \cong H_i(S^{\dim \Delta - \dim F - 1}; \mathbf{k})$ for all i , where S^j is the j -dimensional sphere. Say Δ is a *homology ball* if $H_i(\text{link}_\Delta(F); \mathbf{k})$ vanishes for $i < \dim \Delta - \dim F - 1$ and is isomorphic to either 0 or \mathbf{k} for $i = \dim \Delta - \dim F - 1$. Furthermore, the boundary complex $\partial \Delta$ of Δ , consisting of all faces F for which $H_{\dim \Delta - \dim F - 1}(\text{link}_\Delta(F); \mathbf{k}) = 0$, is a homology sphere (of codimension 1). In particular, simplicial spheres (resp. balls) are homology spheres (resp. balls).

A polytope is *stacked* if it can be obtained from a simplex by repeatedly taking the convex hull with a vertex beyond some facet. A homology sphere is *stacked* if it is combinatorially isomorphic to the boundary complex of a stacked polytope.

2.2. Rigidity. We mostly follow the presentation in Kalai's [13]. Let $G = (V, E)$ be a graph, and $d(a, b)$ denote Euclidean distance between points a and b in Euclidean space. A d -embedding $f : V \rightarrow \mathbf{R}^d$ is called *rigid* if there exists an $\epsilon > 0$ such that if $g : V \rightarrow \mathbf{R}^d$ satisfies $d(f(v), g(v)) < \epsilon$ for every $v \in V$ and $d(g(u), g(w)) = d(f(u), f(w))$ for every $\{u, w\} \in E$, then $d(g(u), g(w)) = d(f(u), f(w))$ for every $u, w \in V$. G is called *generically d -rigid* if the set of its rigid d -embeddings is open and dense in the topological vector space of all of its d -embeddings. Given a d -embedding $f : V \rightarrow \mathbf{R}^d$, a *stress* of f is a function $w : E \rightarrow \mathbf{R}$ such that for every vertex $v \in V$

$$\sum_{u: \{v, u\} \in E} w(\{v, u\})(f(v) - f(u)) = 0.$$

The stresses of f form a vector space, called the *stress space*. Its dimension is the same for generic d -embeddings (namely, for an open and dense set in the space of all d -embeddings of G). A graph G is called *generically d -stress free* if this dimension is zero.

If a generic $f : V \rightarrow \mathbf{R}^d$ is rigid, then $f_1(G) \geq df_0(G) - \binom{d+1}{2}$. Thus, if Δ is a simplicial complex of dimension $d - 1$ whose 1-skeleton is generically d -rigid, then $f_1(\Delta) \geq df_0(\Delta) - \binom{d+1}{2}$, and $g_2(\Delta)$ is the dimension of the stress space of any generic embedding. Based on these observations for Δ the boundary of a simplicial d -polytope with $d \geq 3$, and more general complexes, Kalai [13] extended the LBT and characterized the minimizers.

For a d -polytope P with a simplicial 2-skeleton, the toric $g_2(P)$ equals $g_2(\partial P) := f_1(P) - df_0(P) + \binom{d+1}{2}$, and by a result of Alexandrov (cf. Whiteley [24]), it equals the dimension of the stress space of the 1-skeleton of P .

For our LBT for ASP, we will need the following very special case of Kalai's monotonicity¹, which Kalai proved using rigidity arguments.

Theorem 2.2 (Kalai's Monotonicity [14, Thm. 4.1], weak form). *Let $d \geq 4$, P a d -polytope with a simplicial 2-skeleton, and F a facet of P . Then*

$$g_2(P) \geq g_2(F).$$

Equivalently, $f_1(P) - f_1(F) \geq (df_0(P) - \binom{d+1}{2}) - ((d-1)f_0(F) - \binom{d}{2})$.

3. A LOWER BOUND THEOREM FOR ALMOST SIMPLICIAL POLYTOPES

Recall that a simplicial d -polytope is called *stacked* if it can be obtained from a d -simplex by repeated *stacking*, namely, adding a vertex beyond a facet and taking the convex hull.

¹Kalai's monotonicity conjecture on the toric g -polynomials, asserting that $g(P) \geq g(F)g(P/F)$ coefficientwise for any face F of P , was first proved for rational polytopes by Braden and MacPherson [9]. By the combinatorial intersection homology, it is now known to hold in full generality; cf. [8].

While stacked d -polytopes on n vertices, denoted $S(d, n)$, may have different combinatorial structures, they all have the same f -vector, given by

$$f_k(S(d, n)) = \phi_k(d, n) := \begin{cases} \binom{d}{k}n - \binom{d+1}{k+1}k & \text{for } k = 1, \dots, d-2 \\ (d-1)n - (d+1)(d-2) & \text{for } k = d-1. \end{cases}$$

For any integers $d \geq 3$, $s \geq 0$ and $n \geq d + s + 1$, let F be a stacked $(d-1)$ -polytope with $d + s$ vertices. Construct a pyramid over F and then stack $n - d - s - 1$ times over facets of the resulting polytope which are different from F to obtain a polytope $S(d, n, s)$ in $\mathcal{P}(d, n, s)$. One easily computes the f -vector of $S(d, n, s)$, since refining F by its (unique) stacked triangulation refines the boundary complex of $S(d, n, s)$ to a stacked simplicial sphere with f -vector $f(S(d, n))$. We obtain

$$f(S(d, n, s)) = f(S(d, n)) - (0, 0, \dots, 0, s, s).$$

Note that for $n \geq s + 4$, any $P \in \mathcal{P}(3, n, s)$ has f -vector $f(P) = (1, n, 3n - 6 - s, 2n - 4 - s) = f(S(3, n, s))$. We are ready to state the LBT for ASP; its minimizers will be characterized later (see Theorems 3.2 and 3.4).

Theorem 3.1 (LBT for ASP). *Let $d \geq 3$, $s \geq 0$, $n \geq d + s + 1$. Then for any $P \in \mathcal{P}(d, n, s)$ and $1 \leq i \leq d - 1$ we have*

$$f_i(S(d, n, s)) \leq f_i(P).$$

Proof. We proceed by induction on d , the case $d = 3$ was verified above. Let $d \geq 4$. By a result of Whiteley [24], the 1-skeleton of P is generically d -rigid, hence $f_1(P) \geq \phi_1(d, n)$, and by the MPW reduction, $f_i(P) \geq \phi_i(d, n)$ for all $2 \leq i \leq d - 3$ as well; see [13, Thm. 12.2]².

Denote by (P, F) the ASP-pair, and by $\deg_P(v)$ the degree of a vertex v in the 1-skeleton of P . We now prove the inequality for the facets, by a variation of the MPW reduction. Note that the vertex figure P/v in P of any vertex $v \in \text{vert } F$ is an ASP (with $\deg_P(v)$ vertices), while for any vertex $v \in \text{vert } P \setminus \text{vert } F$ P/v is a simplicial polytope; cf. [10, Thm. 11.5]. Furthermore, for a vertex $v \in \text{vert } F$, letting $s_v := \deg_F(v) - (d - 1) \geq 0$ gives $P/v \in \mathcal{P}(d - 1, \deg_P(v), s_v)$.

²Kalai's theorem contains a typo. It includes the case $i = k$, while it holds only for $i < k$, where P is k -simplicial. Our ASP P is $(d - 2)$ -simplicial.

Double counting the number of pairs (v, A) for a vertex v in a facet A of P , we obtain the following inequalities:

$$\begin{aligned}
d(f_{d-1}(P) - 1) + (d + s) &= \sum_{v \in \text{vert } P} f_{d-2}(\text{link}_P(v)) \\
&\geq \sum_{v \in \text{vert } P \setminus \text{vert } F} ((d-2) \deg_P(v) - d(d-3)) + \sum_{v \in \text{vert } F} ((d-2) \deg_P(v) - d(d-3) - s_v) \\
&= 2(d-2)f_1(P) - d(d-3)f_0(P) - 2f_1(F) + (d-1)(d+s) \\
&\geq 2(d-2) \left[df_0(P) - \binom{d+1}{2} \right] - d(d-3)f_0(P) - 2 \left[(d-1)f_0(F) - \binom{d}{2} \right] + (d-1)(d+s) \\
&= d(d-1)f_0(P) - d(d+1)(d-2) - s(d-1),
\end{aligned}$$

where the first inequality is by the induction hypothesis and the second inequality is by Kalai's monotonicity Theorem 2.2 and the LBT inequality for $f_1(P)$. Comparing the LHS with the RHS gives

$$f_{d-1}(P) \geq \phi_{d-1}(d, n) - s.$$

The inequality for $f_{d-2}(P)$ follows from the inequality for $f_{d-1}(P)$ by double counting. Since any ridge in P is contained in exactly two facets, counting the number of pairs (R, A) for a ridge R in a facet A of P , we obtain that

$$2f_{d-2}(P) = d(f_{d-1}(P) - 1) + f_{d-2}(F).$$

Applying the classical LBT to the simplicial polytope F with $f_0(F) = d + s$, we get

$$2f_{d-2}(P) \geq d(f_{d-1}(P) - 1) + (d-2)(d+s) - d(d-3),$$

and applying the lower bound for $f_{d-1}(P)$ yields, after dividing both sides by 2, the desired lower bound $f_{d-2}(P) \geq \phi_{d-2}(d, n) - s$. \square

We now turn our attention to characterizing the minimizers of Theorem 3.1. We start with some terminology and background.

A proper subset A of the vertices of a d -polytope P is called a *missing k -face* of P if the cardinality of A is $k + 1$, the simplex on A is not a face of P , but for any proper subset B of A the simplex on B is a face of P . If A is a missing $(d-1)$ -face of P then adding the simplex A cuts P into two d -polytopes P_1, P_2 , glued along the simplex A . We denote this operation by $P = P_1 \# P_2$. Repeating this procedure on each P_i until no piece P_i contains a missing $(d-1)$ -face results in a decomposition $P = P_1 \# P_2 \# \cdots \# P_t$, where intersections along missing $(d-1)$ -faces of P define a tree whose vertices are the P_i 's. Note that for $d \geq 3$ a decomposition of P as above is uniquely defined; just insert all the missing $(d-1)$ -faces. Call such a decomposition the *prime decomposition* of P , and call each P_i a *prime factor* of P . Denote by Δ_P the polyhedral complex defined by the prime decomposition of P . Then a simplicial d -polytope P is stacked iff all its prime factors are d -simplices. This

definition immediately extends to polyhedral spheres where the operation $\#$ corresponds to the topological connected sum.

We start with the characterization of the minimizers for the easier case $d > 4$.

Theorem 3.2 (Characterization of minimizers for $d > 4$). *Let $d > 4$ and $P \in \mathcal{P}(d, n, s)$. Let Δ_F be the polyhedral complex corresponding to the prime decomposition of F , and let Δ be the refinement of the boundary complex ∂P of P obtained by refining F by Δ_F . Assume there is some $1 \leq i \leq d - 1$ for which $f_i(P) = f_i(S(d, n, s))$. Then, all prime factors in the prime decomposition of Δ are d -simplices. In particular, $f(P) = f(S(d, n, s))$.*

Remark 3.3. *Let Q be a polytope, G a facet of Q and H the hyperplane containing G . An H -stacking on Q is the operation of (i) adding a new vertex w in H , beyond a facet of G (with respect to G) such that perturbing w from H to the side of the interior of Q makes w beyond a facet of Q , and (ii) taking the convex hull of w and Q . The minimizers P considered in Theorem 3.2 are precisely the polytopes that can be obtained by the following recursive procedure: start with a d -simplex having a facet in a hyperplane H , and repeatedly either H -stack or (usual) stack over a facet not in H .*

Proof of Theorem 3.2. By the MPW reduction and the variation of it we used in the proof of Theorem 3.1, it is enough to consider the case $i = 1$. From Kalai's monotonicity (Theorem 2.2) and our assumption $g_2(P) = 0$, it follows that $g_2(F) = 0$. As F is simplicial of dimension ≥ 4 , Kalai's [13, Thm. 1.1(ii)] says that F is stacked, thus Δ is a simplicial $(d - 1)$ -sphere. Since $g_2(\Delta) = 0$, by [13, Thm. 1.1(ii)] again, Δ is stacked, as desired. In particular, $f(P) = f(S(d, n, s))$. \square

For $d = 4$, F need not be stacked. For example, the pyramid over any simplicial 3-polytope is a minimizer. We obtain the following characterization of minimizers.

Theorem 3.4 (Characterization of minimizers for $d = 4$). *Let $P \in \mathcal{P}(4, n, s)$, and keep the notation of Theorem 3.2. Assume there is some $1 \leq i \leq d - 1$ for which $f_i(P) = f_i(S(d, n, s))$. Then, the prime factors in the prime decomposition of Δ are either d -simplices with no facet contained in $|F|$, or pyramids over prime factors of F .*

In order to prove this theorem we first need to show generic d -rigidity for the 1-skeleton of a much larger class of complexes.

Let \mathcal{C}_k be the family of homology k -balls Δ such that:

- the induced subcomplex $\Delta[I]$ on the set of internal vertices I has a connected 1-skeleton, and
- for any edge e in the boundary complex $\partial\Delta$, there exists a 2-simplex T , $e \subset T$, such that T has a vertex in I .

Note that any homology k -ball Δ whose boundary $\partial\Delta$ is an induced subcomplex is in \mathcal{C}_k . In particular, for $P \in \mathcal{P}(d, n, s)$, the simplicial complex $P' = \partial P - \{F\}$ is in \mathcal{C}_{d-1} .

Lemma 3.5. *Let $d \geq 4$. The 1-skeleton of any $\Delta \in \mathcal{C}_{d-1}$ is generically d -rigid. Thus, $f_1(\Delta) \geq df_0(\Delta) - \binom{d+1}{2}$.*

Proof. The proof is as in Kalai's proof of the classical LBT [13]; for completeness we give a brief description. For any $v \in I$, $\text{link}_\Delta(v)$ is a homology sphere of dimension ≥ 2 , hence its 1-skeleton is generically $(d-1)$ -rigid. Then the Cone Lemma (cf. [13, Sec. 3]) says the 1-skeleton of $\text{star}_\Delta(v)$ is generically d -rigid. From the Gluing Lemma [3] and the first item in the definition of \mathcal{C}_{d-1} , it then ensues that the 1-skeleton G of $\cup_{v \in I} \text{star}_\Delta(v)$ is also generically d -rigid. The definition of \mathcal{C}_{d-1} finally gives that G is the 1-skeleton of Δ . \square

Proof of Theorem 3.4. Consider a prime factor L of Δ . Then L is a 4-polytope with a generically 4-rigid 1-skeleton. As $g_2(P) = 0$, the 1-skeleton of L , denoted by G , must be generically 4-stress free. Thus, $g_2(L) = 0$.

If L does not contain a facet in F , then L is simplicial, with $g_2(L) = 0$, hence is stacked by [13, Thm. 1.1]. Being also prime, L is a 4-simplex.

Assume then that L contains a facet F'' contained in $|F|$, so (L, F'') is an ASP-pair. If L has a unique vertex outside F'' , then L is a pyramid over a prime factor of F and we are done. Assume the contrary, so there is an edge $vu \in G$ with $v, u \notin F''$ (for concreteness, taking v, u to be the highest two vertices of L above the hyperplane of F works).

First we show that vu satisfies the link condition $\text{link}_L(v) \cap \text{link}_L(u) = \text{link}_L(vu)$, which guarantees that contracting the edge vu in the simplicial complex $\partial L - \{F''\}$ results in $\tilde{\Delta} \in \mathcal{C}_3$; see e.g. [21, Prop. 2.4]³. Indeed, if vu fails the link condition it means that vu is contained in a missing face M , with 3 or 4 vertices. Now, M cannot have 4 vertices as L is prime. If $M = vuz$ then uz is an edge of L not in $\text{link}_L(v)$. Since $\text{link}_L(v)$ is a homology 2-sphere (thus, a simplicial 2-sphere), its 1-skeleton is generically 3-rigid. Consequently, the 1-skeleton of $\text{star}_L(v)$ is generically 4-rigid, and adding vu to it yields a 4-stress in G , a contradiction.

Let m be the number of vertices in the cycle $\text{link}_L(vu)$, then $f_1(\tilde{\Delta}) = f_1(L) - m - 1$ and $f_0(\tilde{\Delta}) = f_0(L) - 1$, which implies that $g_2(L) = g_2(\tilde{\Delta}) + (m - 3)$.

If $m > 3$, then applying Lemma 3.5 to $\tilde{\Delta}$ yields $g_2(L) > 0$, a contradiction. So assume $m = 3$.

³To apply [21, Prop. 2.4], phrased for homology spheres, simply cone the boundary of the homology ball Δ to obtain a homology sphere.

Denote by x, y, z the vertices of $\text{link}_L(vu)$. If the triangle $xyz \in L$, then, as L is prime, both tetrahedra $xyzv, xyzu$ are faces of L , so L is the 4-simplex $xyzuv$, a contradiction (as it has a facet F'' in F).

We are left to consider the case $xyz \notin L$. The argument here is inspired by Barnette [4, Thm. 2]. In this case, the 3-ball formed by the join $vu * \partial(xyz)$ is an induced subcomplex of $\partial L - \{F''\}$. Now replace it by $\partial(vu) * xyz$ (this is a bistellar move) to obtain from $\partial L - \{F''\}$ the complex Δ'' . Clearly Δ'' is a homology 3-ball, and any edge on its boundary is part of a 2-simplex with an internal vertex (just take the same one as in $\partial L - \{F''\}$). To show $\Delta'' \in \mathcal{C}_3$ we are left to show that the graph on the internal vertices I of Δ'' is connected. Assume not, namely removing the edge uv disconnects the induced graph on I in $\partial L - \{F''\}$. In particular, $x, y, z \in F''$. But $xyz \notin L$, so xyz is a missing face of F'' , contradicting that F'' is a prime factor of F .

We conclude that $\Delta'' \in \mathcal{C}_3$, thus, by Lemma 3.5, $\Delta'' \cup \{vu\}$ has a nonzero 4-stress, so $g_2(L) > 0$, a contradiction. The proof is then complete. \square

Remark 3.6. *The above shows that, for any $k \geq 3$, the lower bounds of Theorem 3.1 are valid for any complex in \mathcal{C}_k , and the minimizers in \mathcal{C}_k are exactly the complexes $\partial P - \{F\}$ described in Theorems 3.2 and 3.4.*

4. AN UPPER BOUND THEOREM FOR ALMOST SIMPLICIAL POLYTOPES

Throughout this section, we let $P \in \mathcal{P}(d, n, s)$ denote an almost simplicial polytope, (P, F) the ASP-pair, and $P' = \partial P \setminus \{F\}$ the corresponding shellable simplicial $(d-1)$ -ball. Recall that $\partial P' = \partial F$ is an induced subcomplex of P' .

4.1. ASP generalization of cyclic polytopes. The *moment curve* in \mathbf{R}^d is defined by $t \mapsto (t, t^2, \dots, t^d)$ for $t \in \mathbf{R}^d$. We consider related curves $x(t) = (t, t^2, \dots, t^{d-r}, p_1(t), \dots, p_r(t))$, where $p_i(t)$ are continuous functions in t for $i = 1, \dots, r$. Later, a special choice of the curve $x(t)$ and points on it will give our maximizer polytope $C(d, n, s)$.

We let $V(t_1, \dots, t_d)$ denote the *Vandermonde determinant* on variables t_1, \dots, t_d .

$$V(t_1, \dots, t_d) := \begin{vmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_d \\ t_1^2 & t_2^2 & \cdots & t_d^2 \\ \vdots & \vdots & \cdots & \vdots \\ t_1^{d-1} & t_2^{d-1} & \cdots & t_d^{d-1} \end{vmatrix} = \prod_{1 \leq i < j \leq d} (t_j - t_i).$$

Lemma 4.1. *Consider the curve $x(t)$. Then the following holds:*

- (1) *Any $d - r + 1$ points on the curve $x(t)$ are affinely independent.*

- (2) For any n distinct numbers t_1, \dots, t_n , the polytope $Q = \text{conv}(\{x(t_1), \dots, x(t_n)\})$ is $(d - r - 1)$ -simplicial.
- (3) The polytope Q is $\lfloor (d - r)/2 \rfloor$ -neighbourly.

Proof. Consider d real numbers $t_1 < \dots < t_d$, the corresponding points $x(t_i)$, and the matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ t_1 & t_2 & \cdots & t_d & t \\ t_1^2 & t_2^2 & \cdots & t_d^2 & t^2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ t_1^{d-r} & t_2^{d-r} & \cdots & t_d^{d-r} & t^{d-r} \\ p_1(t_1) & p_1(t_2) & \cdots & p_1(t_d) & p_1(t) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ p_r(t_1) & p_r(t_2) & \cdots & p_r(t_d) & p_r(t) \end{pmatrix}.$$

For any $d - r + 1$ points $x(t_{i_1}), \dots, x(t_{i_{d-r+1}})$, the determinant $V(t_{i_1}, \dots, t_{i_{d-r+1}})$ is nonzero, which implies the first assertion. The second assertions follows immediately from the first.

To prove the third assertion proceed as in [12, Sec. 4.7]. Consider a set $S_k = \{x(t_{i_j}) : j = 1, \dots, k\}$, $1 \leq i_j \leq n$, with $k \leq \lfloor (d - r)/2 \rfloor$ and the polynomial

$$\beta(t) = \prod_{i=1}^k (t - t_{i_j})^2 = \beta_0 + \beta_1 t + \cdots + \beta_{2k} t^{2k}.$$

Let $b = (\beta_1, \dots, \beta_{2k}, 0, \dots, 0)$ be a vector in \mathbf{R}^d and $H = \{x \in \mathbf{R}^d : x \cdot b = -\beta_0\}$ a hyperplane in \mathbf{R}^d . Here \cdot denotes the dot product of vectors.

All the points in S_k are clearly contained in H , and for any other $x(t_l) \in \{x(t_1), \dots, x(t_n)\} \setminus S_k$ we have $x(t_l) \cdot b = -\beta_0 + \beta(t_l) > -\beta_0$. Thus, S_k is the vertex set of a simplex face of Q . \square

Consider the curve $y(t) = (t, t^2, \dots, t^{d-1}, p(t))$, where

$$p(t) := (n - 1)^{(t-1)(d-1)} t(t + 1) \cdots (t + d + s - 1),$$

and n and s are fixed integers with $n > d + s$ and $s \geq 0$. Let $C(d, n, s) := \text{conv}(\{y(t_1), \dots, y(t_n)\})$, where $t_i = -s - d + i$ for $i = 1, \dots, n$. Also, let $T = \{t_i : i = 1, \dots, n\}$, $I = \{t_i : i = 1, \dots, d + s\}$ and $y(S) := \{y(t_i) : t_i \in S\}$ for $S \subset T$.

The following proposition collects a number of properties of the d -polytope $C(d, n, s)$.

Proposition 4.2. *The d -polytope $C(d, n, s)$ ($n > d + s$) satisfies the following properties.*

- (1) $C(d, n, s) \in \mathcal{P}(d, n, s)$.
- (2) **Gale's evenness condition:** *A d -subset S_d of $\text{vert } C(d, n, s)$ such that $S_d \not\subset I$ forms a simplex facet iff, for any two elements $u, v \in T \setminus S_d$, the number of elements of S_d between u and v on the curve $y(t)$ is even.*

Proof. (1) We first show that the first $d + s$ vertices span a facet F . Let $y = (y_1, \dots, y_d) \in \mathbf{R}^d$ and let

$$D((t_1, t_2, \dots, t_d); y) := \begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ t_1 & t_2 & \dots & t_d & y_1 \\ t_1^2 & t_2^2 & \dots & t_d^2 & y_2 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ t_1^{d-1} & t_2^{d-1} & \dots & t_d^{d-1} & y_{d-1} \\ p(t_1) & p(t_2) & \dots & p(t_d) & y_d \end{vmatrix}.$$

Let $D(y) := D((t_1, t_2, \dots, t_d); y)$ and consider the hyperplane $H_D := \{y \in \mathbf{R}^d : D(y) = 0\}$. The points $y(t_i)$ ($i = 1, \dots, d + s$) are all contained in H_D , by vanishing of the last row of $D(y(t_i))$. Let $y(t^*) \in \text{vert } C(d, n, s) \setminus y(I)$, then $D(y(t^*)) = p(t^*)V(t_1, \dots, t_d) > 0$ since $p(t^*) > 0$ and $V(t_1, \dots, t_d) > 0$. Thus, F is a facet of $C(d, n, s)$.

We now show that every other facet is a simplex. Consider any $(d + 1)$ -set $\{t_{i_1} < \dots < t_{i_d} < t_{i_{d+1}} = t^*\} \subset T$ not contained in I . Thus, $t^* \in T \setminus I$. Consider the determinant $E(y) := D((t_{i_1}, t_{i_2}, \dots, t_{i_d}); y)$. The hyperplane $H_E := \{y \in \mathbf{R}^d : E(y) = 0\}$ contains all the points $y(t_{i_j})$ ($j = 1, \dots, d$). We need to show that $E(y(t^*)) \neq 0$.

Note that $p(t) = 0$ for $t \in I$ and $p(t) > 0$ for $t \in T \setminus I$. Also, note that $|t_a - t_b| \leq n - 1$ for $t_a, t_b \in [-s - d + 1, -s - d + n]$. For the sake of clarity assume d is odd; the case of even d is analogous. Computing $E(y(t^*))$ by expanding w.r.t. the last row gives

$$\begin{aligned} & (p(t^*)V(t_{i_1}, \dots, t_{i_d}) - p(t_{i_d})V(t_{i_1}, \dots, t_{i_{d-1}}, t^*)) + \dots \\ & + (p(t_{i_2})V(t_{i_1}, t_{i_3}, \dots, t^*) - p(t_{i_1})V(t_{i_2}, \dots, t^*)). \end{aligned}$$

The definition of $p(t)$ implies that each pair-summand is nonnegative and the first pair-summand is positive, and so the determinant is positive. Indeed, for $j > 1$, if $p(t_{i_j}) = 0$ then also $p(t_{i_{j-1}}) = 0$ and the corresponding pair-summand vanishes. Otherwise, let $V(j) := V(t_{i_1}, \dots, t_{i_{j-1}}, t_{i_{j+1}}, \dots, t_{i_{d+1}})$ for short. Then,

$$p(t_{i_j})V(j) \geq (n - 1)^{(d-1)(t_{i_j}-1)} \prod_{l=0}^{d+s-1} (t_{i_{j-1}} + l) \frac{V(j-1)}{(n-1)^{d-1}} \geq p(t_{i_{j-1}})V(j-1).$$

This completes the proof of the first assertion.

(2) Consider a set $S_d = \{t_{i_1} < \dots < t_{i_d}\} \not\subset I$. Let $t^* \in T$, $t_{i_{j-1}} < t^* < t_{i_j}$ (include also the cases $t^* < t_{i_1}$ with $j = 1$ and $t_{i_d} < t^*$ where we put $j = d + 1$). From the above reasoning we see that if the column $y(t^*)$ in the determinant $E(y(t^*))$ is placed between the columns $y(t_{i_{j-1}})$ and $y(t_{i_j})$ then the resulting determinant is positive. To achieve this, we swap $d - j + 1$ times the column $y(t^*)$, which gives that the sign of $E(y(t^*))$ is $(-1)^{d-j+1}$. Consequently, on the curve $y(t)$, between $[-s - d + 1, -s - d + n]$, the determinant $E(y(t^*))$ changes sign whenever the variable passes through one of the values t_{i_j} ($i = 1, \dots, d$), and we are done. \square

A polytope $C(d, n, s)$ will be called *almost cyclic*. Having established in Lemma 4.1 that $C(d, n, s)$ is $\lfloor (d - 1)/2 \rfloor$ -neighbourly, we can compute its h -vector, in steps. Recall that $P' = \partial P \setminus \{F\}$.

Proposition 4.3. *Let $P \in \mathcal{P}(d, n, s)$ be $\lfloor (d - 1)/2 \rfloor$ -neighbourly, and (P, F) the ASP-pair. Then,*

$$\begin{aligned} h_k(P') &= \binom{n - d - 1 + k}{k}, & \text{if } 0 \leq k \leq \lfloor (d - 1)/2 \rfloor; \\ h_{d-k}(P') &= \binom{n - d - 1 + k}{k} - \binom{s + k - 1}{k}, & \text{if } 1 \leq k \leq \lfloor (d - 1)/2 \rfloor. \end{aligned}$$

Proof. First note that $f_{k-1}(P') = \binom{n}{k}$ for $k \leq \lfloor (d - 1)/2 \rfloor$. Thus, it ensues that

$$h_k(P') = \sum_{i=0}^k (-1)^{k-i} \binom{d-i}{k-i} \binom{n}{i} = \binom{n - d - 1 + k}{k}.$$

We now consider the remaining values of k . Using that F is a neighborly simplicial $(d - 1)$ -polytope we obtain that $g_k(F) = \binom{d+s-(d-1)+k-2}{k}$, for $0 \leq k \leq \lfloor (d - 1)/2 \rfloor$. Thus, from Eq. (2) of Proposition 2.1 it follows, for $1 \leq k \leq \lfloor (d - 1)/2 \rfloor$, that

$$h_{d-k}(P') = \binom{n - d - 1 + k}{k} - \binom{s + k - 1}{k}.$$

\square

Observe that, for even d , being $\lfloor (d - 1)/2 \rfloor$ -neighbourly does not determine the value of $h_{d/2}(P')$. With the help of Gale's evenness condition we can compute the number of facets of $C(d, n, s)$, and together with Proposition 4.3 and Eq. (1), we can compute $h_{d/2}(C(d, n, s))$ for any even d as well. We let $C' := C(d, n, s) - \{F\}$.

Proposition 4.4. *For the ASP-pair $(C(d, n, s), F)$ with d even, consider the simplicial ball C' . Then*

$$f_{d-1}(C') = \left(\binom{n-d/2-1}{d/2} + \sum_{i=0}^{d/2-1} 2 \binom{n-d-1+i}{i} \right) - \binom{s+d/2}{d/2}.$$

Proof. The counting argument for the facets of C' , based on Gale evenness, goes as in the proof of the number of facets of cyclic polytopes (cf. [25, Cor. 8.28]), with the difference that we discard the Gale d -tuples formed solely by the first $d+s$ vertices, thus we discard exactly $\binom{s+d/2}{d/2}$ of them. \square

Corollary 4.5. *The h -numbers of C' are given by:*

$$\begin{aligned} h_k(C') &= \binom{n-d-1+k}{k}, & \text{if } 0 \leq k \leq \lfloor (d-1)/2 \rfloor; \\ h_{d-k}(C') &= \binom{n-d-1+k}{k} - \binom{s+k-1}{k}, & \text{if } 1 \leq k \leq \lfloor d/2 \rfloor. \end{aligned}$$

Proof. The case of odd d was already established by Proposition 4.3 since $C(d, n, s)$ is $\lfloor (d-1)/2 \rfloor$ -neighbourly. For the case of even d it remains to compute $h_{d/2}(P)$. Equating the corresponding expression in Proposition 4.4 with the expression of f_{d-1} in Eq. (1), after substituting the known values of h_k for $k \neq d/2$, gives

$$\begin{aligned} h_{d/2}(C') &= \binom{n-d/2-1}{d/2} + \sum_{i=0}^{d/2-1} \binom{s+i-1}{i} - \binom{s+d/2}{d/2} \\ &= \binom{n-d/2-1}{d/2} - \binom{s+d/2-1}{d/2}, \end{aligned}$$

as desired. \square

4.2. An upper bound theorem for almost simplicial polytopes. We are now in a position to state an upper bound theorem for almost simplicial polytopes $P \in \mathcal{P}(d, n, s)$.

Theorem 4.6 (UBT for ASP). *Any almost simplicial polytope $P \in \mathcal{P}(d, n, s)$ satisfies*

$$(3) \quad h_k(P') \leq \binom{n-d-1+k}{k}, \quad \text{if } 0 \leq k \leq \lfloor (d-1)/2 \rfloor;$$

$$(4) \quad h_{d-k}(P') \leq \binom{n-d-1+k}{k} - \binom{s+k-1}{k}, \quad \text{if } 1 \leq k \leq \lfloor d/2 \rfloor.$$

Thus,

$$f_{i-1}(P) \leq f_{i-1}(C(d, n, s)) \quad \text{for } i = 1, 2, \dots, d,$$

for the almost cyclic d -polytope $C(d, n, s)$. Equality for some f_{i-1} with $\lfloor (d-1)/2 \rfloor \leq i \leq d$ implies that P is $\lfloor (d-1)/2 \rfloor$ -neighbourly.

Proof of Theorem 4.6 via [2, Thm. 3.10]. The inequalities on $h_k(P')$ hold for $0 \leq k \leq d-1$ by [2, Thm. 3.10], as P' is a special case of a homology ball whose boundary is an induced subcomplex. From Corollary 4.5 and Eq. (1) the inequality $f_{i-1}(P) \leq f_{i-1}(C(d, n, s))$ follows. Equality for some f_{i-1} with $d \geq i \geq \lfloor (d-1)/2 \rfloor$ implies, by Eq. (1), the equality $h_k(P') = \binom{n-d-1+k}{k}$ for all $0 \leq k \leq \lfloor (d-1)/2 \rfloor$, and thus, again by Eq. (1), that P is $\lfloor (d-1)/2 \rfloor$ -neighbourly. \square

Remark 4.7 (More maximizers.). *As is the case with neighborly polytopes, we expect that there are many combinatorially distinct ASPs achieving the upper bounds in the UBT for ASP. Here we sketch another such construction, based on a certain perturbation of the Cayley polytope constructed by Karavelas and Tzanaki [15, Sec.5]: there, two neighborly $(d-1)$ -polytopes P_1 and P_2 are placed in parallel hyperplanes in \mathbf{R}^d so that the Cayley polytope $P = \text{conv}(P_1 \cup P_2)$ is $\lfloor \frac{d-1}{2} \rfloor$ -neighborly and all $\lfloor d/2 \rfloor$ -subsets with a vertex in P_1 and a vertex in P_2 form faces of P . Let $n = f_0(P)$ and $d+s = f_0(P_2)$. Thus, for d odd, any small enough perturbation of the vertices of P_1 into general position will change P , by considering the new convex hull, into $Q \in \mathcal{P}(d, n, s)$ ((Q, P_2) is the ASP-pair) with $f(Q) = f(C(d, n, s))$; so Q is a maximizer. For d even, in order for Q to be a maximizer, we need the small perturbation be such that all $d/2$ -subsets of $\text{vert}(P_1)$ become faces of Q . To achieve this, we recall more from the construction of [15], and make a variation: consider the images of the functions $\gamma_1(t, z_1, z_2) = (t, z_1 t^{d-1}, t^2, t^3, \dots, t^{d-2}, z_2 t^d) \subset \mathbf{R}^d$ and $\gamma_2(t, z_1) = (z_1 t^{d-1}, t, t^2, \dots, t^{d-2}, -1) \subset \mathbf{R}^{d-1} \times \{-1\} \subset \mathbf{R}^d$. The vertices of P_1 (resp. P_2) are on appropriate locations on the curve $\gamma_1(t, z_1^*, 0) \subset \mathbf{R}^{d-1} \times \{0\}$ (resp. $\gamma_2(t, z_1^*)$), for small enough fixed $z_1^* > 0$. It is possible to show, by appropriate determinant computation, that for a small enough fixed $z_2^* > 0$, perturbing the vertices $\gamma_1(t_i, z_1^*, 0)$ of P_1 to $\gamma_1(t_i, z_1^*, z_2^*)$ makes all $d/2$ -subsets of P_1 faces of Q ; so Q is a maximizer.*

We proceed by producing an alternative and elementary proof of the UBT for ASP, via shelling. This will take the rest of this section. Our proof follows ideas from the proof of the classical UBT by McMullen, cf. [25, Sec.8.4], and from a recent work of Karavelas and Tzanaki [15]. The key new ingredient is Lemma 4.9 below, for which we need some preparation.

Let $P \in \mathcal{P}(d, n, s)$ and (P, F) an ASP-pair. Let Q be a polytope obtained from P by stacking a new vertex y beyond F . The d -polytope Q is simplicial. The set of proper faces of Q is the disjoint union of the faces of the complex $P' := \partial P - \{F\}$ and the faces of Q which contain y . Consequently, for all $k \geq 0$,

$$(5) \quad h_k(Q) = h_k(P') + h_{k-1}(F).$$

Recall the star $\text{star}_{\mathcal{C}}(F)$ of a face F in a polytopal complex \mathcal{C} is the polytopal subcomplex generated by all the faces of \mathcal{C} containing F .

We will use a line shelling of Q with some special properties:

Lemma 4.8. *Let (P, F) be an ASP-pair, and $v \in F$ a vertex. Then we can choose the aforementioned vertex y beyond F such that, for (P, F, y, Q) as above, there is a line shelling of Q which shells the star of y first and then proceeds to shell the rest of the star of v .*

Proof. For an oriented line ℓ that shells P , with F being first followed by the rest of the facets in the star of v (cf. [25, Thm. 8.12, Cor. 8.13]⁴), place y on ℓ beyond F to make Q . Now perturb ℓ to obtain the desired line shelling. \square

Consider any vertex $v \in \text{vert } Q$ and let $S(Q)$ be a shelling of the facets of Q . Then, clearly,

- The restriction of $S(Q)$ to $\text{star}_Q(v)$ yields a shelling of $\text{star}_Q(v)$ (cf. [25, Lem. 8.7]); denote it by $S_v(Q)$.
- A shelling of $\text{star}_Q(v)$ induces a shelling of $\text{link}_Q(v)$ by deleting v from the facets.
- Since $\partial F = \text{link}_Q(y)$, it follows that $S(Q)$ induces a shelling $S(F)$ of F .
- Recursively, $S(F)$ induces a shelling of $\text{link}_F(v)$ if $v \in \text{vert } F$.

Now consider any vertex $v \in F$, a shelling $S(Q)$ as guaranteed in Lemma 4.8, and the induced shellings $S(Q/v)$ of Q/v , $S(F)$ of F and $S(F/v)$ of F/v .

Following [15, Sec. 4], call the facet F_j of Q *active* if it is the new facet to be added to the shelling process $S(Q)$. Let $F_j|F$ be the active facet of $S(F)$ which is the restriction of F_j to F (if $y \in F_j$). Let F_j/v be the active facet of $S(Q/v)$ induced by F_j (if $v \in F_j$), $F_j|F/v$ be the active facet of $S(F/v)$ induced by $F_j|F$ (if $vy \subset F_j$), and $F_j|v$ be the active facet of $S_v(Q)$ induced by F_j (if $v \in F_j$); so $F_j|v = \{v\} \cup F_j/v$ in this case. Let $R_j \subseteq F_j$, $R_j/v \subseteq F_j/v$, $R_j|F \subseteq F_j|F$, $R_j|F/v \subseteq F_j|F/v$, and $R_j|v$ be the corresponding new minimal faces in the shellings $S(Q)$, $S(Q/v)$, $S(F)$, $S(F/v)$, and $S_v(Q)$ respectively.

Finally, let $h_k^j(Q)$ denote the value of h_k up to step j , namely $h_k(\cup_{i \leq j} F_i)$, and similarly for the other complexes.

The following key lemma allows us to relate the difference in h -numbers along a shelling of Q and F to that of Q/v and F/v .

Lemma 4.9. *For any vertex $v \in \text{vert } F$ and at any step j of the shelling $S(Q)$, we have, for all $k \geq 0$, that*

$$h_k^j(Q) - h_k^j(Q/v) \geq h_k^j(F) - h_k^j(F/v).$$

Proof. While shelling $\text{star}_Q(y)$, the minimal face R_j of F_j in $S(Q)$ and the minimal face $R_j|F$ of $F_j|F$ in $S(F)$ coincide at every step, since $F = Q/y$ and $S(Q)$ shells the star of y

⁴Here we use the extension of the notion of shellability to polyhedral complexes.

first. Therefore, while shelling $\text{star}_Q(y)$, for all $k \geq 0$, it follows that

$$h_k^j(Q) = h_k^j(F).$$

For the same reason, if $F_j \in \text{star}_Q(v) \cap \text{star}_Q(y)$, then, regardless of whether $v \in R_j$ or $v \notin R_j$, we have, for all $k \geq 0$, that

$$h_k^j(Q/v) = h_k^j(F/v).$$

Thus, while shelling $\text{star}_Q(y)$, it ensues for all $k \geq 0$ that

$$h_k^j(Q) - h_k^j(Q/v) = h_k^j(F) - h_k^j(F/v).$$

After the shelling has left $\text{star}_Q(y)$, we get no new contributions to $h_k(F)$ or $h_k(F/v)$ for all $k \geq 0$, so the RHS does not change.

After shelling $\text{star}_Q(y)$ and while still shelling $\text{star}_Q(v)$, we have that the minimal faces R_j and R_j/v of $S(Q)$ and $S(Q/v)$, respectively, coincide, so the LHS does not change either. To see that $R_j = R_j/v$, first note that $R_j/v \subseteq R_j$ (as the complex at the j step of $S(Q)$ contains the complex at the j step of $S_v(Q)$). We show the reverse containment. Assume by contradiction that there is a facet F'' of F_j which is in the subcomplex $\cup_{i < j} F_i$ of Q but not in the subcomplex $\cup_{i < j, v \in F_i} F_i$ of $\text{star}_Q(v)$. As we have already left $\text{star}_Q(y)$, $y \notin F_j$ so F'' is a facet of F . However, also $v \in F$, so we must have $v \in F''$, as otherwise, by convexity, $|F_j| \subset |F|$, a contradiction. But then the (unique) facet F_i in $\text{star}_Q(y)$ containing F'' is also in $\text{star}_Q(v)$, a contradiction.

Thus, for all $k \geq 0$,

$$h_k^j(Q) - h_k^j(Q/v) = h_k^j(F) - h_k^j(F/v).$$

After the shelling has left $\text{star}(v, Q)$ we may get new contributions to $h_k(Q)$ but not any more to $h_k(Q/v)$. This concludes the proof of the lemma. \square

Proposition 4.10. *Let $P \in \mathcal{P}(d, n, s)$ and (P, F) be an ASP-pair. Then, for all $k \geq 0$, we have*

$$h_{d-(k+1)}(P') \leq \frac{n-d+k}{k+1} h_{d-k}(P') + \frac{n-(d+s)}{k+1} g_k(F).$$

Equivalently, for all $k \geq 0$, we have

$$h_{k+1}(P') \leq \frac{n-d+k}{k+1} h_k(P') - \frac{s+k}{k+1} g_k(F) + g_{k+1}(F).$$

Proof. The second inequality follows from the first by the Dehn-Sommerville relations (2). For the first inequality, first we have the following sequence of equalities.

$$\begin{aligned}
 \sum_{v \in \text{vert } Q} h_k(Q/v) &= (k+1)h_{k+1}(Q) + (d-k)h_k(Q) \\
 &= (k+1)h_{d-(k+1)}(Q) + (d-k)h_{d-k}(Q) \\
 &= (k+1) \left(h_{d-(k+1)}(P') + h_{d-(k+1)-1}(F) \right) + (d-k)h_{d-k}(P') \\
 &\quad + (d-k)h_{d-k-1}(F) \\
 &= \left((k+1)h_{d-(k+1)}(P') + (d-k)h_{d-k}(P') \right) + (k+1)h_{k+1}(F) \\
 &\quad + (d-1-k)h_k(F) + h_k(F) \\
 (6) \quad &= (k+1)h_{d-(k+1)}(P') + (d-k)h_{d-k}(P') + h_k(F) + \sum_{v \in \text{vert } F} h_k(F/v).
 \end{aligned}$$

For the first equality, see, e.g., [25, Eq. 8.27a], while for the second, use Dehn-Sommerville Eq. (2) for Q . The third equality follows from Eq. (5), the forth from Eq. (2) again, this time for F , and the last equality follows from [25, Eq. 8.27a] again.

As $F \cong Q/y$, Eq. (6) then becomes

$$(7) \quad \sum_{v \in \text{vert } P' \setminus \text{vert } F} h_k(Q/v) + \sum_{v \in \text{vert } F} (h_k(Q/v) - h_k(F/v)) = (k+1)h_{d-(k+1)}(P') + (d-k)h_{d-k}(P').$$

From Lemma 4.9 and the fact that any vertex $v \in \text{vert } P' \setminus \text{vert } F$ has the same link in both Q and P' , it then follows that

$$\begin{aligned}
 \sum_{v \in \text{vert } P' \setminus \text{vert } F} h_k(Q/v) + \sum_{v \in \text{vert } F} (h_k(Q/v) - h_k(F/v)) &\leq \sum_{v \in \text{vert } P' \setminus \text{vert } F} h_k(P'/v) \\
 (8) \quad &\quad + \sum_{v \in \text{vert } F} (h_k(Q) - h_k(F)).
 \end{aligned}$$

A shelling of P' that shells $\text{star}_{P'}(v)$ first shows that $h_k(P'/v) \leq h_k(P')$ for all $k \geq 0$. Consequently, from Eqs. (7) and (8) it follows that

$$\begin{aligned}
 (k+1)h_{d-(k+1)}(P') + (d-k)h_{d-k}(P') &\leq \sum_{v \in \text{vert } P' \setminus \text{vert } F} h_k(P') + \sum_{v \in \text{vert } F} (h_k(Q) - h_k(F)) \\
 &= (n - (d+s))h_k(P') + (d+s)(h_k(Q) - h_k(F)) \\
 &= nh_k(P') + (d+s)(h_{k-1}(F) - h_k(F)) \\
 &= n(h_{d-k}(P') + g_k(F)) - (d+s)g_k(F).
 \end{aligned}$$

The last two equations ensue from Eq. (5) and Eq. (2), respectively. Hence, the desired inequality follows. \square

We are now in a position to prove the inequalities of Theorem 4.6 using the shelling approach.

Proof of Theorem 4.6 via Shellings. The inequalities Eq. (3) for P' in Theorem 4.6 hold for any Cohen-Macaulay complex, and can also be proved exactly as in [25, Lem. 8.26]. We prove Eq. (4) by induction on k . The case $k = 1$ holds with equality by Eq. (2): $h_{d-1}(P') = n - d - s$. Suppose now that the inequality Eq. (4) holds for $k - 1 \leq \lfloor d/2 \rfloor - 1$. By Proposition 4.10,

$$h_{d-k}(P') \leq \frac{n-d+k-1}{k} \left(\binom{n-d-1+k-1}{k-1} - \binom{s+k-2}{k-1} \right) + \frac{n-(d+s)}{k} g_{k-1}(F).$$

From the application of the g -theorem⁵, cf. [25, Cor. 8.38], to F , it follows that $g_{k-1}(F) \leq \binom{d+s-(d-1)+k-3}{k-1}$ for $0 \leq k-1 \leq \lfloor d/2 \rfloor - 1$. Thus, the previous inequality becomes

$$\begin{aligned} h_{d-k}(P') &\leq \frac{n-d+k-1}{k} \left(\binom{n-d-1+k-1}{k-1} - \binom{s+k-2}{k-1} \right) + \frac{n-(d+s)}{k} \binom{s+k-2}{k-1} \\ &= \binom{n-d-1+k}{k} - \binom{s+k-1}{k}, \end{aligned}$$

as desired. \square

5. A GENERALIZED LOWER BOUND THEOREM FOR ALMOST SIMPLICIAL POLYTOPES

Recall that $P' = \partial P - \{F\}$. We find it convenient to state the results here using the *toric g -vector* of P (rather than the simplicial h -vectors of P' and ∂F ; both are possible). The toric h - and g -vectors were introduced by Stanley [23] and the reader is referred there for further background. They are defined by a polynomial recurrence

$$h(P, t) = \sum_{0 \leq i \leq \dim P} h_i(P) t^i := \sum_{F < P} g(F, t) (t-1)^{\dim(P) - \dim(F) - 1},$$

where the sum runs over all proper faces F of the polytope P , setting $g(P, t) = \sum_{0 \leq i \leq \lfloor d/2 \rfloor} g_i(P) t^i$, $g_i(P) = h_i(P) - h_{i-1}(P)$ as in the simplicial case, and the initial conditions $h(\emptyset, t) = 1 = g(\emptyset, t)$. Stanley showed that $h(P, t)$ is a palindrome. Karu [16] showed that for any polytope P , $g(P, t)$ is nonnegative coefficientwise, by proving a hard Lefschetz theorem for the combinatorial intersection homology module associated with P .

The structure of polytopes with $g_2 = 0$ is not known, cf. [14, Conj. 11,12], however, the results of Section 3 do provide a characterization for almost simplicial polytopes, similar to the simplicial case. In this section, we extend these results to higher g -numbers, similar to the classical generalized lower bound theorem (GLBT); see [19, 20].

Recall that a simplicial d -polytope P is $(r-1)$ -stacked if the simplicial complex $P(d-r)$, formed by all subsets of $\text{vert } P$ whose $(d-r+1)$ -subsets are faces of ∂P , triangulates P . Similarly, extend this notion to any geometric triangulation of a polytope boundary.

⁵In fact, this consequence follows easily just from the fact that ∂F is a Cohen-Macaulay complex.

Theorem 5.1. *Let $P \in \mathcal{P}(d, n, s)$ with $g_k(P) = 0$ for some $1 \leq k < d/2$, and let (P, F) be an ASP-pair. Then F is $(k - 1)$ -stacked. Furthermore, let P'' be the simplicial complex refining ∂P by replacing F by its $(k - 1)$ -stacked triangulation $F(d - 1 - k)$. Then $P''(d - k)$ triangulates P .*

Proof. By Kalai's monotonicity, cf.[8], $g_k(F) = 0$, so from the GLBT [20] it follows that $F(d - 1 - k)$ triangulates F .

By a very recent result of Adiprasito [1, Theorem 7.2], $F(d - 1 - k)$ is regular, so lifting it by a small enough amount into convex position gives that P'' is isomorphic to the boundary complex of a simplicial d -polytope Q . Further, as $g_k(P'') = 0$ holds⁶, the GLBT applied to Q gives that Q is $(k - 1)$ -stacked, thus $P''(d - k)$ triangulates P . \square

Problem 5.2. *What can be said in the case $2 < k = d/2$? (Compare with Theorem 3.4 for $2 = k = d/2$.)*

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⁶This can be shown directly from the definition, or by noting that the intersection homology modules of the complete fans of P and of P'' coincide up to degree k .

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